Trigonometry 1

1. Prove that $\frac{\sin\theta\tan\theta}{\tan\theta-\sin\theta} = \frac{\tan\theta+\sin\theta}{\sin\theta\tan\theta}$

Consider:

$(\tan\theta + \sin\theta)(\tan\theta - \sin\theta) - (\sin\theta\tan\theta)^2$

$= tan^2\,\theta - sin^2\,\theta - sin^2\,\theta tan^2\,\theta$

$= tan^2\,\theta(1 - sin^2\,\theta) - sin^2\,\theta$

$= tan^2\,\theta cos^2\,\theta - sin^2\,\theta$

$= \frac{sin^2\,\theta}{cos^2\,\theta} cos^2\,\theta - sin^2\,\theta = 0$

Rearrange, we get $\frac{\sin\theta\tan\theta}{\tan\theta-\sin\theta} = \frac{\tan\theta+\sin\theta}{\sin\theta\tan\theta}$.

2. Given : $-\sqrt{3}\cos 2x + \sin 2x = R\sin(2x + \alpha)$, find R and α in degrees.

$\sin 2x - \sqrt{3}\cos 2x = 2\left[\frac{1}{2}\sin 2x - \frac{\sqrt{3}}{2}\cos 2x\right] = 2[\sin 2x\cos 60° - \cos 2x\sin 60°]$

$= 2\sin(2x - 60°)$

$\therefore R = 2, \alpha = -60°.$

3. By completing the square, find the greatest and least values, as θ varies, of $cos^2\theta - \cos\theta + 6$.

$y = cos^2\theta - \cos\theta + 6 = \left(\cos\theta - \frac{1}{2}\right)^2 + \frac{23}{4}$

When $\cos\theta = \frac{1}{2}$, $y_{min} = \frac{23}{4}$

The maximum value occurs when $\cos\theta = -1$, $y_{max} = \left(-1 - \frac{1}{2}\right)^2 + \frac{23}{4} = 8$

4. Solve $2\cos(60° + \theta) + 2\sin(30° + \theta) = \sqrt{3}$ where $-180° < \theta < 180°$

$2(\cos 60°\cos\theta - \sin 60°\sin\theta) + 2(\sin 30°\cos\theta + \cos 30°\sin\theta) = \sqrt{3}$

$2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) + 2\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) = \sqrt{3}$

$\therefore \cos\theta = \frac{\sqrt{3}}{2}$

The general solution is $\theta = 360°n \pm 30°$, where n is an integer.
Since $-180° < \theta < 180°$, the solution within the range is $\theta = 360°(0) \pm 30°$ or $\theta = \pm 30°$.

5. Proof: $\sin(\alpha+\beta)\sin(\alpha-\beta) = cos^2\beta - cos^2\alpha$

$$\sin(\alpha+\beta)\sin(\alpha-\beta) = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = (1-\cos^2\alpha)\cos^2\beta - \cos^2\alpha\,(1-\cos^2\beta)$$
$$= cos^2\beta - cos^2\alpha$$

6. Solve the equation $\frac{\sqrt{5}}{2}\sec\theta - \tan\theta = 2$ for $0° \le \theta \le 360°$.

$$\frac{\sqrt{5}}{2}\sec\theta - \tan\theta = 2 \Rightarrow \frac{\sqrt{5}}{2}\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = 2 \Rightarrow \sqrt{5} - 2\sin\theta = 4\cos\theta \Rightarrow 4\cos\theta + 2\sin\theta = \sqrt{5}$$

$$\frac{2}{\sqrt{5}}\cos\theta + \frac{1}{\sqrt{5}}\sin\theta = \frac{1}{2} \ldots (1)$$

Put $\cos\alpha = \frac{2}{\sqrt{5}}$, then $\sin\alpha = \frac{1}{\sqrt{5}}$, $\alpha \approx 26.565051177078°$

(1) becomes $\cos\theta\cos\alpha + \sin\theta\sin\alpha = \frac{1}{2} \Rightarrow \cos(\theta-\alpha) = \frac{1}{2}$

The general solution $\theta - \alpha = 360°n \pm 60°$,
$\therefore \theta \approx 360°n \pm 60° + 26.565051177078°$, where n is an integer.
Since $0° \le \theta \le 360°$, $\theta \approx 360°(0) + 60° + 26.565051177078° \approx \mathbf{86.6°}$
Or $\theta \approx 360°(1) - 60° + 26.565051177078° \approx \mathbf{326.6°}$

7. If $tan\,x = 2tan\,y$, show that $tan(x+y) = \frac{3\,sin\,2y}{3\,cos\,2y-1}$.

$$tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y} = \frac{2tan\,y + \tan y}{1-2tan\,y\tan y} = \frac{3tan\,y}{1-2tan^2\,y} = \frac{3\frac{sin\,y}{cos\,y}}{1-2\frac{sin^2\,y}{cos^2\,y}}$$

$$= \frac{3\sin y\,\cos y}{cos^2\,y - 2sin^2\,y} = \frac{3(2\sin y\,\cos y)}{2cos^2\,y - 4sin^2\,y}$$

$$= \frac{3(2\sin y\,\cos y)}{3(cos^2\,y - sin^2\,y) - (cos^2\,y + sin^2\,y)} = \frac{3\,sin\,2y}{3\,cos\,2y-1}$$

8. Solve $-\sqrt{3}\cos 2x + \sin 2x = 1$ for general solution in terms of degrees.

$\sqrt{3}\cos 2x - \sin 2x = -1$

$\frac{\sqrt{3}}{2}\cos 2x - \frac{1}{2}\sin 2x = -\frac{1}{2}$

$\cos 2x \cos 30° - \sin 2x \sin 30° = -\frac{1}{2}$

$\cos(2x + 30°) = -\frac{1}{2}$

$2x + 30° = 360°n \pm 120°$, where n is an integer.

$2x = 360°n \pm 120° - 30°$

$x = 180°n \pm 60° - 15°$

$\therefore x = 180°n + 45°$ or $180°n - 75°$, where n is an integer.

9. Prove $\frac{1-\sin\theta}{1+\sin\theta} \equiv (\sec\theta - \tan\theta)^2$

R.H.S. $= \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 = \frac{(1-\sin\theta)^2}{\cos^2\theta} = \frac{(1-\sin\theta)^2}{1-\sin^2\theta} = \frac{(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$ = L.H.S.

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